4. **The Thermodynamic Helmholtz Resonator** 热力学亥姆霍兹共振器

(a) [1] Determine the fundamental frequency of a sound wave resonating in a cylindrical pipe of length H that is **closed at one end and open at the other,** as shown in Fig. 4-1. Given that the speed of sound is $v = 343\ \mathrm{m/s}$ *and* $H = 30\ cm$, calculate the fundamental frequency f.

(a) [1] 确定在一个长度为 H、一端封闭另一端开口的圆柱形管中（如图 4-1 所示）产生声波共振的基频。已知声速 $v = 343\ \mathrm{m/s}$，管长 $H = 30\ \mathrm{cm}$，计算该基频 f。

(b) However, the result from part (a) deviates significantly from actual measurements. Therefore, a more accurate model is required. We model the bottle as a Helmholtz resonator (see Fig. 4-2). A Helmholtz resonator consists of a cavity of volume V connected to the open atmosphere (pressure P_0) by a cylindrical neck of length L and radius r. The air in the neck acts as a piston of mass m, causing adiabatic compression and expansion of the air inside the cavity. The adiabatic speed of sound is given by $v = \sqrt{\frac{\gamma RT}{M}}$, where M is the molar mass of air, γ is the adiabatic index, R is the ideal gas constant, and T is the temperature.Starting from the adiabatic condition for the air in the cavity, we can show that for small volume changes ΔV, the pressure change ΔP is related to the displacement x of the air plug by $\Delta P = -kx$. Hence, the equation of motion for the air in the neck can be derived and we can show that the air column will oscillate. Assume the mass of the air in the neck is $m = \rho_0 \pi r^2 L$, where ρ_0 is the equilibrium air density.

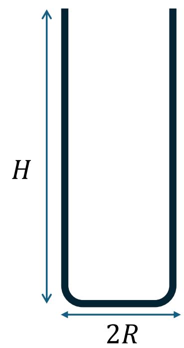

Fig 4-1

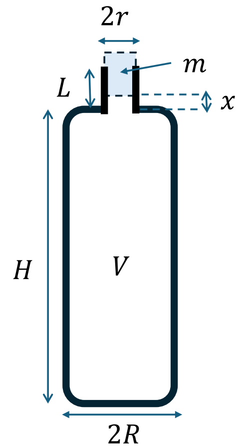

Fig 4-2

(b) 然而，(a) 部分的结果与瓶子的实际测量值偏差较大。因此，我们需要一个更精确的模型来描述瓶子产生的声波。我们将瓶子视为一个亥姆霍兹共振器（Helmholtz resonator，见图 4-2）。亥姆霍兹共振器由体积为 V 的空腔通过长为 L、半径为 r 的圆柱形颈管与大气（压强 P_0）相连而成。腔内空气经历绝热压缩和膨胀，而颈管内的空气则视作质量为 m 的活塞。绝热声速由 $v = \sqrt{\frac{\gamma RT}{M}}$ 给出，其中 M 为空气摩尔质量，γ 为绝热指数，R 为理想气体常数，T 为温度。从腔内空气的绝热条件出发，可证明对于微小的体积变化 ΔV，腔内压强变化 ΔP 与颈管内空气柱的位移 x 满足关系：$\Delta P = -kx$。由此，可推导颈管内空气的运动方程，並试证明颈管内的空气将作振荡运动。设颈管内空气质量为 $m = \rho_0 \pi r^2 L$，其中 ρ_0 为空气的平衡密度。

(bi) [5] Find the frequency f of the oscillating air. Express the answer in terms of P_0, ρ_0, r, L, V and γ.

(bi) [5] 求振动空气的频率 f。请用 P_0, ρ_0, r, L, V 和 γ 来表达你的答案。

(bii) [1] In reality, the length L is replaced by the effective length $L_{\text{eff}} \approx L + 1.7r$, accounting for the air mass outside the neck that also oscillates. Calculate the fundamental frequency produced by the bottle with the following parameters: $H = 23\ \text{cm}, \mathrm{R} = 3.5\ \text{cm},\ L = 7\ \text{cm}, r = 1\ \text{cm}$.

(bii) [1] 在现实中，长度 L 会被有效长度 $L_{eff} \approx L + 1.7r$ 所取代，以考虑瓶颈外部同样发生振动的空气质量。计算具有以下参数的水瓶所产生的基频：$H = 23\ \text{cm}$，$R = 3.5\ \text{cm}$，$L = 7\ \text{cm}$，$r = 1\ \text{cm}$。

(c) The water bottle is constructed from a material with a linear thermal expansion coefficient α. Suppose the resonator is calibrated to a frequency f_0 at a reference temperature T_0. When the ambient temperature increases to $T = T_0 + \Delta T$ (where $\Delta T \ll T_0$):

(ci) [2] Account for both the change in the speed of sound of the air and the thermal expansion of the bottle's dimensions. Derive an expression for the new resonance frequency f in terms of $f_0, \Delta T, T_0$ and α.

(cii) [1] Determine the "critical" thermal expansion coefficient α_c at which the frequency of the bottle remains constant regardless of small temperature fluctuations. Express α_c in terms of $f_0, \Delta T, T_0$.

(c) 该水瓶由线性热膨胀系数为 α 的材料制成。假设在参考温度 T_0 下，共振器的校准频率为 f_0。当环境温度升高至 $T = T_0 + \Delta T$（其中 $\Delta T \ll T_0$）时：

(ci) [2] 推导新共振频率 f 的表达式，用$f_0, \Delta T, T_0$ 和 α 表达。

(cii) [1] 确定一个“临界”热膨胀系数 α_c，使得在温度发生微小波动时，瓶子的频率保持不变。用 $f_0, \Delta T, T_0$ 表达 α_c。

Solution:

(a)

$$f = \frac{v}{\lambda} = \frac{4v}{H} \Rightarrow f = 286 \text{ Hz.}$$

(b) [4 marks]

$$PV^{\gamma} = c \Rightarrow \ln P + \gamma \ln V = c \Rightarrow \frac{dP}{P} + \gamma \frac{dV}{V} = 0$$

Notice that $dV = \pi r^2 x$ and the force on the air in the neck is $F = AdP$,

$$m\ddot{x} = \pi r^2 dP = -\gamma \pi r^2 \frac{P_0}{V} dV = -\frac{\gamma \pi^2 r^4 P_0}{V} x = -kx$$

The oscillating frequency is

$$\omega^2 = \frac{k}{m} = \frac{\gamma \pi^2 r^4 P_0}{\pi r^2 L \rho_0 V} = \frac{\gamma \pi r^2 P_0}{L \rho_0 V}$$

$$\Rightarrow \omega = \sqrt{\frac{\gamma P_0}{\rho_0} \times \frac{\pi r^2}{VL}}$$

And the fundamental frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \times \sqrt{\frac{\gamma P_0}{\rho_0} \times \frac{\pi r^2}{VL}} \rightarrow \frac{v}{2\pi} \sqrt{\frac{\pi r^2}{VL_{eff}}} \approx 110\, Hz$$

(c) [6 marks]

$$f \propto v \sqrt{\frac{r^2}{VL}}$$

Since $v \propto \sqrt{T}$,

$$f \propto \sqrt{\left(1 + \frac{\Delta T}{T_0}\right)\left(\frac{1}{1 + \alpha \Delta T}\right)^2} \propto \left(1 + \frac{\Delta T}{2T_0}\right)(1 - \alpha \Delta T) = 1 + \left(\frac{1}{2T_0} - \alpha\right)\Delta T$$

$$f = \left(1 + \left(\frac{1}{2T_0} - \alpha\right)\Delta T\right) f_0$$

Cii

$$\alpha_c = \frac{1}{2T_0}$$