3. Transmission Line Modelling 传输线模型

Consider a high-frequency signal propagating along a transmission line. The line is modeled as a series of infinitesimal segments of length Δx (See Fig 3-1). Let L and R be the series inductance and resistance per unit length, and G and C be the shunt conductance and capacitance per unit length, respectively. Assume L, R, G, and C are frequency-independent constants.

考虑高频信号在传输线中的传播。我们将传输线建模为长度为 Δx 的无穷小段的集合 (见图 3-1)。令 L 和 R 分别为单位长度的串联电感和电阻，G 和 C 分别为单位长度的并联电导和电容。假设 L, R, G, C 均为与频率无关的常数。

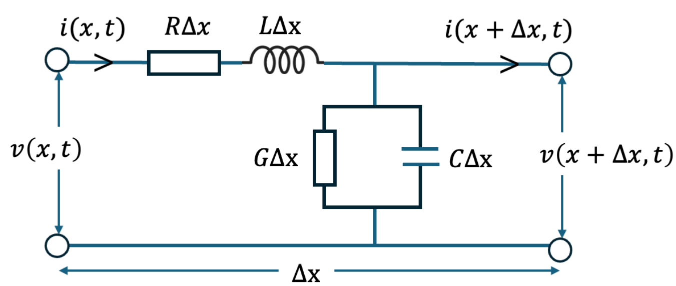

Fig 3-1

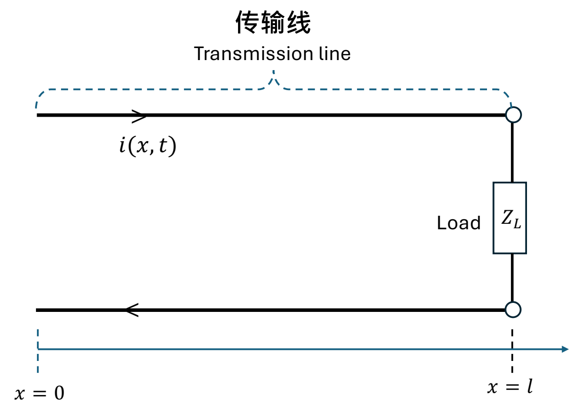

Fig 3-2

(ai) [4] By applying Kirchhoff's laws to the infinitesimal circuit model as shown in Fig. 3-1, derive the two coupled partial differential equations relating current $i(x,t)$ and voltage $v(x,t)$ as $\Delta x \to 0$.

(ai) [4] 对图 3-1 的电路模型应用基尔霍夫定律，推导出当 $\Delta x \to 0$ 时，描述电流 $i(x,t)$ 与电压 $v(x,t)$ 关系的两个耦合偏微分方程。

(aii) [1] For a sinusoidal signal of angular frequency ω, define $i(x,t) = \mathrm{Re}(I(x)e^{j\omega t})$ and $v(x,t) = \mathrm{Re}(V(x)e^{j\omega t})$, the equations reduce to:

$$\frac{d^2 I(x)}{dx^2} = \gamma^2 I(x), \quad \frac{d^2 V(x)}{dx^2} = \gamma^2 V(x)$$

Find the complex propagation constant γ^2 in terms of L, R, G, C, and ω. Here $j = \sqrt{-1}$ is the imaginary number and Re(z) is the real part of a complex number z.

(aii) [1] 对于角频率为 ω 的正弦信号，设 $i(x,t) = \mathrm{Re}(I(x)e^{j\omega t})$ 且 $v(x,t) = \mathrm{Re}(V(x)e^{j\omega t})$，方程可化简为：

$$\frac{d^2 I(x)}{dx^2} = \gamma^2 I(x), \quad \frac{d^2 V(x)}{dx^2} = \gamma^2 V(x)$$

求复传播常数 γ^2 关于 L, R, G, C 和 ω 的表达式。这里 $j = \sqrt{-1}$ 是虚数，Re(z)表示复数 z 的实部。

(aiii) [1] The general solutions are $I(x) = I_0^+ e^{-\gamma x} + I_0^- e^{\gamma x}$ and $V(x) = V_0^+ e^{-\gamma x} + V_0^- e^{\gamma x}$. Define the characteristic impedance $Z_0 = \left|\frac{V_0^+}{I_0^+}\right| = \left|\frac{V_0^-}{I_0^-}\right|$. Determine Z_0 in terms of L, R, G, C, and ω.

(aiii) [1] 已知通解为 $I(x) = I_0^+ e^{-\gamma x} + I_0^- e^{\gamma x}$ 和 $V(x) = V_0^+ e^{-\gamma x} + V_0^- e^{\gamma x}$。定义特性阻抗为 $Z_0 = \left|\frac{V_0^+}{I_0^+}\right| = \left|\frac{V_0^-}{I_0^-}\right|$。求 Z_0 的表达式，用L, R, G, C 和 ω 表达。

(aiv) [2] Find the condition relating L, R, G and C that allows a signal to propagate without distortion. Determine the phase velocity v_p in this case.

(aiv) [2] 试求当 L, R, G, C 满足何种关系时，信号可实现无失真传播？并求出该情形下的相速度 v_p。

(bi) [1] In Fig. 3-2, a transmission line is terminated by a load impedance Z_L. When an incident signal V_i travels along the line, a portion is reflected as V_r. Derive the voltage reflection coefficient, $\Gamma = \frac{V_r}{V_i}$, in terms of Z_L and the characteristic impedance Z_0.

(bi) [1] 如图 3-2 所示，传输线终端接有负载阻抗 Z_L。当入射信号 V_i 沿传输线传播时，部分信号将被反射，形成反射波 V_r。试推导电压反射系数 $\Gamma = \frac{V_r}{V_i}$ 的表达式（用 Z_L 和特性阻抗 Z_0 表示）。

(bii) [1] State the required condition for the incident voltage signal V_i to undergo a π phase change upon reflection.

(bii) [1] 若要使入射电压信号 V_i 在反射时产生 π 的相位变化，负载阻抗 Z_L 需满足什么条件？

Solution

(ai) Applying Kirchhoff's voltage law,

$$v(x,t) - R\Delta x\, i(x,t) - L\Delta x \frac{\partial i(x,t)}{\partial t} - v(x+\Delta x, t) = 0$$

$$\Rightarrow -\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t}$$

By the current law,

$$i(x,t) - G\,\Delta x\, v(x+\Delta x,t) - C\,\Delta x \frac{\partial v(x+\Delta x,t)}{\partial t} - i(x+\Delta x,t) = 0$$

$$\Rightarrow -\frac{\partial i}{\partial x} = Gv + C\frac{\partial v}{\partial t}$$

(aii)

$$-\frac{dV}{dx} = (R + j\omega L)I \qquad (1)$$

$$-\frac{dI}{dx} = (G + j\omega C)V. \qquad (2)$$

$$\Rightarrow \frac{d^2V}{dx^2} = (R+j\omega L)(G+j\omega C)V = \gamma^2 V$$

where

$$\gamma^2 = (R+j\omega L)(G+j\omega C)$$

(aiii)

$$I(x) = I_0^+ e^{-\gamma x} - I_0^- e^{\gamma x}, \qquad V(x) = V_0^+ e^{-\gamma x} + V_0^- e^{\gamma x}$$

Sub. Into Eqtn (1) and (2),

$$\gamma(I_0^+ e^{-\gamma x} + I_0^- e^{\gamma x}) = (G+j\omega C)(V_0^+ e^{-\gamma x} + V_0^- e^{\gamma})$$

For the equation to be satisfied for all x, we have

$$\Rightarrow Z_0 = \frac{V_0^+}{I_0^+} = \frac{V_0^-}{I_0^-} = \frac{\gamma}{G+j\omega C} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

Similarly,

(aiv) For a sinusoidal signal, the voltage has the form

$$v(x,t) = V(x)e^{j\omega t} = \left(V_0^+\, e^{-(\alpha+j\beta)x} + V_0^- e^{(\alpha+j\beta)x}\right)e^{j\omega t}$$

The phase velocity is given by

$$v_p = \frac{\omega}{\beta}$$

$$\gamma = (R^2+\omega^2L^2)^{\frac{1}{4}}(G^2+\omega^2C^2)^{\frac{1}{4}}e^{\frac{i(\theta_1+\theta_2)}{2}}$$

$$\Rightarrow \beta = (R^2+\omega^2L^2)^{\frac{1}{4}}(G^2+\omega^2C^2)^{\frac{1}{4}}\sin\frac{(\theta_1+\theta_2)}{2}$$

$$\tan\theta_1 = \frac{\omega L}{R}, \qquad \tan\theta_2 = \frac{\omega C}{G}$$

If

$$\frac{R}{L} = \frac{G}{C} = k$$

$$\Rightarrow \gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{LC}\sqrt{\left(\frac{R}{L}+j\omega\right)\left(\frac{G}{C}+j\omega\right)} = \sqrt{LC}(k+j\omega)$$

$$\Rightarrow \alpha = \sqrt{\frac{R}{L}}\sqrt{LC} = \sqrt{RC}$$

$$\beta = \omega\sqrt{LC}$$

$$\Rightarrow v_p = \frac{1}{\sqrt{LC}}$$

Remark: $\frac{R}{L} = \frac{G}{C}$ is the distortionless condition, even though the signal gets weaker (attenuation), the **shape** of the pulse remains intact because every frequency component travels at the same speed and dies out at the same rate. This was the key to making long-distance telegraphy and telephony possible.

(bi) Reflection Coefficient

The boundary condition at the load requires $v(l,t) = i(l,t)Z_L$. Since $V(l) = V^+e^{-\gamma l} + V^-e^{\gamma l}$ and $I(l) = I^+e^{-\gamma l} - I^-e^{\gamma l} = \frac{V^+e^{-\gamma l} - V^-e^{\gamma l}}{Z_0}$

$$V^+e^{-\gamma l} + V^-e^{\gamma l} = \frac{Z_L}{Z_0}(V^+e^{-\gamma l} - V^-e^{\gamma l})$$
$$\Rightarrow \Gamma_L = \frac{V^-e^{\gamma l}}{V^+e^{-\gamma l}} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

(bii) To gain a π phase, $\Gamma = -1, \Rightarrow \Gamma_L < \Gamma_0$.