2. [10] Rods and Planes 棒与面

(a) Gravitational Field Generated by Rods 由细棒产生的引力场

(ai) [1] Find the magnitude of gravitational field at a distance d from an infinitely long thin rod with mass per unit length λ.

(aii) [3] Consider 2026 infinitely long thin rods oriented along the z axis, each with mass per unit length λ. The 2026 rods are on positions $(x_n, y_n) = \left(r_0 \cos\frac{2n\pi}{2026}, r_0 \sin\frac{2n\pi}{2026}\right)$, where $0 \le n \le 2025$. Find the net force on a point mass with mass m located at $(d, 0)$. (Hint: You can consider the complex roots of the polynomial: $(z+d)^{2026} - r_0^{2026} = 0$)

(aiii) [1] Suppose the small point mass is constrained to only move along the x axis, determine if it will oscillate around the origin. (Just answer yes or no)

(ai) [1] 求距离一根线质量密度为 λ 的"無限长细棒"距离 d 处的引力场强度。

(aii) [3] 考虑 2026 根沿 z 方向排列的無限长细棒，每根细棒的质量线密度皆为 λ。第 n 根细棒在平面上的位置为 $(x_n, y_n) = \left(r_0 \cos\frac{2n\pi}{2026}, r_0 \sin\frac{2n\pi}{2026}\right)$。求质量为 m 的质点在 $(d, 0)$ 点的合外力。

（提示：你可以考虑多项式$(z+d)^{2026} - r_0^{2026} = 0$ 的复根）

(aiii) [1] 若该质点仅被限制在 x 轴运动，判斷其会否在原点附近來回移动。（只需回答"是"或"否"）

(b) Gravitational Field Generated by a Plate. 由平板产生的引力场

(bi) [1] Consider an infinite horizontal plane with mass per unit area σ, find the magnitude of gravitational field a distance h above the plate.

(bii) [2] Consider a flat and horizontal circular plate with radius R and total mass m. Find the magnitude of gravitational field due to the plate at a distance $h = R$ above the center of the plate if the plate has uniform density.

(Hint: $\int_0^1 \frac{x\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} = 1 - \frac{\sqrt{2}}{2}$)

(biii) [2] Consider a horizontal, equilateral triangular plate of mass m, and side length a. Find the magnitude of gravitational field due to the plate at a distance $h = \frac{\sqrt{6}a}{12}$ above the centroid of the plate if the plate has uniform density. (Hint: The distance from the center of a regular tetrahedron to any of its face is $\frac{\sqrt{6}}{12}$ of the side length of each face.)

(bi) [1] 考虑一张无限大的水平平板，其面质量密度为 σ，求在距离平板高度为 h 处的引力场强度。

(bii) [2] 考虑一张水平放置的圆形平板，半径为 R，质量平均分佈，总质量为 m。求在距离平板中心高度$h = R$处，由该平板产生的引力场强度。（提示：$\int_0^1 \frac{x\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} = 1 - \frac{\sqrt{2}}{2}$）

(biii) [2] 考虑一个质量为 m、边长为 a的水平正三角形板。求位于平板质心正上方、高度 $h = \frac{\sqrt{6}a}{12}$ 处的引力场强度。（提示：正四面体的中心到任一面的距离为其边长的$\frac{\sqrt{6}}{12}$。）

Solution:

(ai) By using Gauss' Law of gravitation, we can easily get

$$g = \frac{2G\lambda}{d}$$

(aii) Note that the G force caused by the rods are given by $F_n = \frac{2Gm\lambda}{d_n}\widehat{d_n}$, where $\overrightarrow{d_n}$ is the position vector of the rod from the test charge.

Therefore, we can interpret each d_n as a complex number, where we know it will be a root of the equation $(z+d)^{2026} - r_0^{2026} = 0$.

Denote the complex conjugate of d_n as d_n^*, note that $d_n = d_{2026-n}^*$ for $n \ne 0$, and for $n = 0$, $d_0 = d_0^*$.

Notice how $\frac{2Gm\lambda}{d_n^*}$ has the argument same as d_n and absolute value $\frac{2Gm\lambda}{|d_n|}$, which is a perfect complex analogy for the gravitational force caused by a rod at d_n. Therefore, we have

$$\sum_{n=0}^{2025} \frac{1}{d_n^*} = \sum_{n=1}^{2026} \frac{1}{d_{2026-n}} = \sum_{n=0}^{2025} \frac{1}{d_n} = \frac{d_0 d_1 \dots d_{2023} + \dots + d_1 d_2 \dots d_{2024}}{d_0 d_1 \dots d_{2024}} \tag{1}$$

By considering the coefficients of the polynomial equation, we get

$$\sum_{n=0}^{2025} \frac{1}{d_n} = \frac{2026 x_0^{2025}}{r_0^{2026} - x_0^{2026}} \tag{2}$$

Therefore, the total force is

$$F = \sum_{n=0}^{2025} F_n = 2Gm\lambda \sum_{n=0}^{2025} \frac{1}{d_n} = \frac{4052 Gm\lambda x_0^{2025}}{r_0^{2026} - x_0^{2026}} \tag{3}$$

The force only has component parallel to the x axis by symmetry, so a positive value here indicates a force towards the positive x direction.

(Detailed solution) By the Gauss' law, the gravitational force on the test mass is

$$\vec{F}_n = -\frac{2Gm\lambda}{d_n}\hat{d}_n = -2Gm\lambda\left(\frac{d_x}{d^2}\hat{x} + \frac{d_y}{d^2}\hat{y}\right)$$

where $\vec{d}_n$ is the position vector of the rod from the point mass and $\hat{d}_n$ is the unit vector.

$$\vec{d}_n = \left(r_0\cos\frac{2n\pi}{2025} - x_0\right)\hat{x} + r_0\sin\frac{2n\pi}{2025}\hat{y}$$

Define the complex number $\tilde{d}_n$:

$$\tilde{d}_n = \left(r_0\cos\frac{2n\pi}{2025} - x_0\right) + ir_0\sin\frac{2n\pi}{2025}$$

$$\Rightarrow F_x = -2Gm\lambda\frac{d_x}{d^2}, \qquad F_y = -2Gm\lambda\frac{d_y}{d^2}$$

$$\tilde{F}_n = -2Gm\lambda\left(\frac{d_x}{d^2} + i\frac{d_y}{d^2}\right) = -2Gm\lambda\frac{1}{d_x - id_y} = -2Gm\lambda\frac{1}{\tilde{d}_n^*}$$

The net complex force can be written as

$$\tilde{F}_{net} = -2Gm\lambda\sum_n\frac{1}{\tilde{d}_n^*}$$

Notice that $\{d_n\}$ are the complex roots of the 2026th order polynomial:

$$(z+d)^{2026} - r_0^{2026} = (z-\tilde{d}_0)(z-\tilde{d}_1)\ldots(z-\tilde{d}_{2025}) = 0$$

$$\Rightarrow (-\tilde{d}_0)(-\tilde{d}_1)\cdots(-\tilde{d}_{2025}) = x_0^{2026} - r_0^{2026}$$
$$\Rightarrow \tilde{d}_0\tilde{d}_1\ldots\tilde{d}_{2025} = x_0^{2026} - r_0^{2026}$$

The coefficient of the linear term z:

$$2026x_0^{2025} = -\left(\tilde{d}_0\tilde{d}_1\ldots\tilde{d}_{2024} + \cdots + \tilde{d}_1\tilde{d}_2\ldots\tilde{d}_{2025}\right)$$

$$\sum_{n=0}^{2025}\frac{1}{\tilde{d}_n} = \frac{\tilde{d}_0\tilde{d}_1\ldots\tilde{d}_{2024} + \cdots + \tilde{d}_1\tilde{d}_2\ldots\tilde{d}_{2025}}{\tilde{d}_0\tilde{d}_1\ldots\tilde{d}_{2025}} \tag{1}$$

By considering the coefficients of the polynomial equation, we get

$$\sum_{n=0}^{2025}\frac{1}{\tilde{d}_n} = \frac{2026x_0^{2025}}{r_0^{2026} - x_0^{2026}} = \sum_{n=0}^{2025}\frac{1}{\tilde{d}_n^*} \tag{2}$$

Therefore, the total force (on the x direction by symmetry) is

$$\vec{F} = Re(\tilde{F})\hat{\mathrm{x}} + \mathrm{Im}(\tilde{F})\hat{\mathrm{y}} = \sum_{n=0}^{2025}F_n\,\hat{\mathrm{x}} = \hat{\mathrm{x}}2Gm\lambda\sum_{n=0}^{2025}\frac{1}{d_n} = \frac{4052Gm\lambda x_0^{2025}}{r_0^{2026} - x_0^{2026}}\hat{\mathrm{x}} \tag{3}$$

(aiii) Since $F \approx \frac{4052Gm\lambda x_0^{2025}}{r_0^{2026}} \geq 0$ for $x > 0$ (positive displacement), the force always points towards the positive x direction for a displacement towards the positive x direction, so there is no oscillation possible.

(bi) It is easy to tell that the force is independent of h and always $g = 2\pi G\sigma$ by Gauss' law.
(bii) We can use integration to solve it, decompose the circular plate into rings.
For a thin ring of thickness dr and radius r, by symmetry, we know the gravitational field is pointing in vertical direction, so

$$\mathrm{d}g = \frac{2Gmhr\mathrm{d}r}{R^2(h^2+r^2)^{\frac{3}{2}}}$$

Doing integration from $r = 0$ to $r = R$ gives

$$g = \int_0^R\frac{2Gmhr}{R^2(h^2+r^2)^{\frac{3}{2}}}\mathrm{d}r = \frac{2Gm}{R^2}\int_0^1\frac{x\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} = \frac{(2-\sqrt{2})Gm}{R^2}$$

(biii) Consider putting a test mass of mass M at a distance $h = \frac{\sqrt{6}a}{12}$ above the centroid of the plate. By symmetry, we can just consider vertical force. The magnitude of upward gravitational force on the plate by the test mass is

$$F = \int_A \overrightarrow{g_M}(\vec{r}) \cdot \hat{n} dm$$

Where $\overrightarrow{g_M}$ is the gravitational field by the point mass at some location on the plate, $\vec{r}$.

Observe that $dm = \frac{m}{A} dA$ and $\int_A \overrightarrow{g_M}(\vec{r}) \cdot \hat{n} dA$ is just the gravitational flux through the plate. By the symmetry of the tetrahedron, we have

$$\int_A \overrightarrow{g_M}(\vec{r}) \cdot \hat{n} dA = \frac{1}{4}(4\pi GM) = \pi GM$$

Therefore, the total gravitational force is $F = \frac{\pi GMm}{A}$

By Newton's third law, the gravitational force on the test mass by the plate has the exact same magnitude

$$F' = \frac{\pi GMm}{A}$$

Therefore, the magnitude of gravitational field due to the plate at a distance $h = \frac{\sqrt{6}a}{12}$ above the centroid of the plate is just

$$g' = \frac{F'}{M} = \frac{\pi Gm}{A} = \frac{4\sqrt{3}\pi Gm}{3a^2}$$