1. Rotating Cylinder 旋转圆柱体

(a) [1] Alice shows Bob her new cylinder collection. One of them is a hollow cylinder. The cylinder has length L, radius r and wall thickness $\delta \ll r$ for both the bases and the curved surface. The mass density of the cylinder is ρ. Find the mass m of the cylinder and the moment of inertia I of the cylinder about its axis.

(a) [1] Alice 向 Bob 展示了她的新圆柱体收藏。其中一个是**空心圆柱体**。该圆柱体的长度为 L，半径为 r，底面和侧面的壁厚均为 $\delta \ll r$。圆柱体的质量密度为 ρ。求圆柱体的质量 m 以及圆柱体绕其轴线的转动惯量 I。

(b) [2] The cylinder is made of carbon nanotubes and has high tensile strength, $\sigma = \frac{F_{max}}{A}$, where F_{max} is the maximum tension force the cylinder can sustain without breaking and A is the cross-sectional area perpendicular to the direction the force is applied (see Fig. 1-1). By considering an element of the cylinder when spinning (refer to the top view of the spinning cylinder in Fig 1-2), estimate the maximum angular velocity ω_{max} at which the cylinder can spin before breaking. Express in terms of σ, ρ and r. Give a numerical value of ω_{max} for $L = 1m$, $r = 10cm$, $\delta = 50\mu m$, $\sigma = 20GPa$, $\rho = 1.3\ \mathrm{g/cm^3}$. Ignore deformation. You may assume the sides break before the bases do.

(b) [2] 该圆柱体由纳米碳管制成，具有极高的抗拉强度 $\sigma = \frac{F_{max}}{A}$，其中 F_{max} 是圆柱体在断裂前能承受的最大拉力，A 是垂直于受力方向的截面积 (見图 1-1)。通过考虑旋转圆柱体的一个微元（参考旋转圆柱体的俯视图 1-2），估算圆柱体断裂前的最大角速度 ω_{max}。用σ, ρ, r 来表达。

给出以下数值：$L = 1\ \mathrm{m}$, $r = 10\ \mathrm{cm}$, $\delta = 50\ \mu\mathrm{m}$, $\sigma = 20\ \mathrm{GPa}$, $\rho = 1.3 \times 9\ \mathrm{g/cm^3}$，计算$\omega_{max}$的数值。忽略形变。你可以假设侧面先于底面断裂。

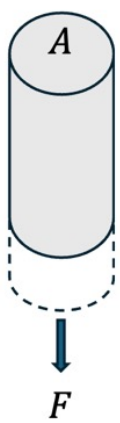
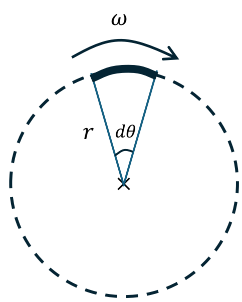


Fig 1-1



Top view
俯视图
Fig 1-2

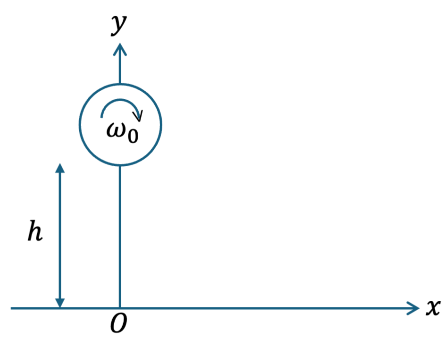


Fig 1-3

In what follows, we shall assume $\omega \ll \omega_{max}$.

(c) [2] In Fig 1-3, Alice uses a motor positioned at a height h above the ground to set the cylinder spinning with an initial angular velocity ω_0. In a prank, Bob uses his "psychic powers" to release the cylinder from the motor without altering its angular velocity. The cylinder, starting with a negligible initial translational velocity, falls toward the ground, which has a coefficient of friction μ. Ignoring air resistance, assume that collisions with the ground are perfectly elastic in the y-direction and that the collision duration t_c is negligible ($gt_c^2 \ll h$), where g represents the acceleration due to gravity. Sketch a qualitative graph representing the trajectory of the cylinder's center of mass.

在下文中，我们假设 $\omega \ll \omega_{max}$。

(c) [2] 在图 1-3，Alice 使用安装在距离地面高度为 h 处的电机，使圆柱体以初始角速度 ω_0旋转。随后 Bob 搞了个恶作剧，利用他的“超能力”使圆柱体脱离电机，且脱离过程不改变其角速度。在圆柱体的初始平动速度可忽略不计的情况下向地面下落，地面与圆柱体间的摩擦系数为 μ。忽略空气阻力，并假设：与地面的碰撞在 y 方向上为完全弹性碰撞，且碰撞时间 t_c 极短（满足 $gt_c^2 \ll h$），其中 g 为重力加速度。请画出圆柱体质心运动轨迹的定性图。

(d) The cylinder hits the ground at O at $t = 0$.
(di) [2] Find the velocity (v_x, v_y) and the angular velocity ω of the cylinder right after the 1st collision.
(dii) [3] Find the position of the ball $x(t)$ and $y(t)$, as well as the angular velocity $\omega(t)$. Express your answers in terms of the number of collisions with the ground N (We take $N = 1$ at $t = 0$), the time elapsed since the last collision Δt, h, g, r, and the ratio $K = \frac{mr^2}{I}$. You can use the floor function $\lfloor x \rfloor$ which denotes the integer part of x (e.g. $\lfloor \pi \rfloor = 3, \lfloor e \rfloor = 2$).

(d) 圆柱体在 $t = 0$ 时击中地面原点 O。

(di) [2] 求第一次碰撞后瞬间圆柱体的速度 (v_x, v_y)和角速度 ω。

(dii) [3] 求圆柱体的位置 $x(t)$和 $y(t)$，以及角速度 $\omega(t)$。用与地面碰撞的次数N（我们取 $t = 0$时 $N = 1$）、自上次碰撞以来经过的时间 Δt、h、g、r 以及比率 $K = \frac{mr^2}{I}$ 来表达你的答案。你可以使用下取整函数（floor function）$\lfloor x \rfloor$，它表示 x 的整数部分（例如 $\lfloor \pi \rfloor = 3, \lfloor e \rfloor = 2$）。

Solution: (Thanks Edison Fu for proposing this problem)

a. $m = 2\pi\rho\delta rL + 2\pi\rho\delta r^2 = 2\pi\rho\delta r(L + r)$
We can divide the moment of inertia calculation into two parts: One for the curved surface and one for the base. The moment of inertia is simply the sum of all individual I's as it is an additive property.

$$I = (2\pi\rho\delta rL)r^2 + 2 * \frac{1}{2}(\pi\rho\delta r^2)r^2 = \pi\rho\delta r^3(2L + r)$$

b. Look at a segment of the cylinder of angle $d\theta$ as measured from the axis. Assume constant tension T throughout the cylinder.

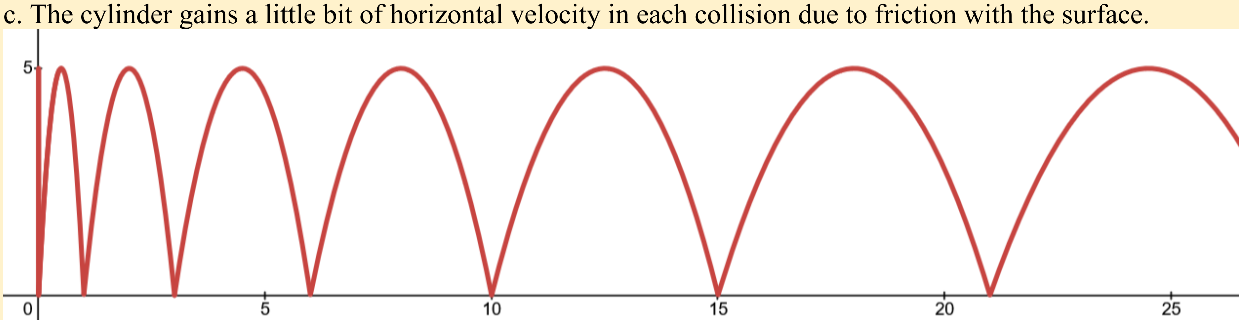
$$A = L\delta, m = \rho L\delta r d\theta$$
$$F = Td\theta = m\omega^2 r$$
$$T = \rho L\delta r^2\omega^2 < \sigma L\delta$$

$$\omega_{max} = \sqrt{\frac{\sigma}{\rho r^2}} = \frac{39000rad}{s}$$

Remark: we will give full credit if you substitute the value of $\rho = 1.3 \times 9\ g/cm^3$ and get $\omega_{max} = 13000\ rad/s$

This is a very large angular velocity. For reference, a helicopter blade spins at about 800rpm which is about 84rad/s.

c. The cylinder gains a little bit of horizontal velocity in each collision due to friction with the surface.



The candidate must realise that the distance between successive bounces initially increases to get any marks on this problem. If they correctly notice this, but make any of the following mistakes, 1 mark may be deducted:
- The height decreases noticeably with more bounces
- The first part of the trajectory is not a vertical line
- The trajectory is not parabolic between bounces (But bad drawing skills is excusable!)

d. $y(t)$ is trivial: It is simply equal to $\sqrt{2gh}\Delta t - \frac{1}{2}g\Delta t^2$.
It is easier to solve for $x(t)$ by considering $v_x(t)$ and $\omega(t)$ at the same time. In each collision, a vertical impulse of $2m\sqrt{2gh}$ is imparted, so the maximum horizontal impulse due to friction is $2\mu m\sqrt{2gh}$.
[1 mark]
As a result, we have $v_x = \left(2\mu\sqrt{2gh}\right)N$ and $\omega r = \omega_0 r - K\left(2\mu\sqrt{2gh}\right)N$ initially. When $N \geq \frac{\omega_0 r}{2\mu(1+K)\sqrt{2gh}}$, the contact point no

longer has relative velocity to the ground as $v_x = \omega r = \frac{\omega_0 r}{1+K}$. After this point, ω and v_x no longer change. [2 marks for correct $\omega(t)$]

$$\omega = \omega_0 - \frac{K}{r}\left(2\mu\sqrt{2gh}\right)N, \quad N < \left\lfloor \frac{\omega_0 r}{2\mu(1+K)\sqrt{2gh}} \right\rfloor = N_0$$

$$\omega = \frac{\omega_0}{1+K}, \quad N \geq N_0$$

In what follows, it will be more convenient if we let $N_0 = \left\lfloor \frac{\omega_0 r}{2\mu(1+K)\sqrt{2gh}} \right\rfloor$.

The time between successive collisions is $2\sqrt{\frac{2h}{g}}$. The contribution to the horizontal distance for each N before steady state is $8\mu h N$. Summing them produces a triangular series.

$$x = \sum_{N'=0}^{N-1} (8\mu h N') + \left(2\mu\sqrt{2gh}\right)N\Delta t = 4\mu h N(N-1) + \left(2\mu\sqrt{2gh}\right)N\Delta t \text{ when } N < N_0$$

$$x = 4\mu h N_0(N_0 - 1) + \frac{\omega_0 r}{1+K}\left(2\sqrt{\frac{2h}{g}}(N - N_0) + \Delta t\right) \text{ when } N \geq N_0$$