1. Rotating Cylinder 旋转圆柱体

(a) [1] Alice shows Bob her new cylinder collection. One of them is a hollow cylinder. The cylinder has length L, radius r and wall thickness $\delta \ll r$ for both the bases and the curved surface. The mass density of the cylinder is ρ. Find the mass m of the cylinder and the moment of inertia I of the cylinder about its axis.

(a) [1] Alice 向 Bob 展示了她的新圆柱体收藏。其中一个是**空心圆柱体**。该圆柱体的长度为 L，半径为 r，底面和侧面的壁厚均为 $\delta \ll r$。圆柱体的质量密度为 ρ。求圆柱体的质量 m 以及圆柱体绕其轴线的转动惯量 I。

(b) [2] The cylinder is made of carbon nanotubes and has high tensile strength, $\sigma = \frac{F_{max}}{A}$, where F_{max} is the maximum tension force the cylinder can sustain without breaking and A is the cross-sectional area perpendicular to the direction the force is applied (see Fig. 1-1). By considering an element of the cylinder when spinning (refer to the top view of the spinning cylinder in Fig 1-2), estimate the maximum angular velocity ω_{max} at which the cylinder can spin before breaking. Express in terms of σ, ρ and r. Give a numerical value of ω_{max} for $L = 1m$, $r = 10cm$, $\delta = 50\mu m$, $\sigma = 20GPa$, $\rho = 1.3g/cm^3$. Ignore deformation. You may assume the sides break before the bases do.

(b) [2] 该圆柱体由纳米碳管制成，具有极高的抗拉强度 $\sigma = \frac{F_{max}}{A}$，其中 F_{max} 是圆柱体在断裂前能承受的最大拉力，A 是垂直于受力方向的截面积 (見图 1-1)。通过考虑旋转圆柱体的一个微元（参考旋转圆柱体的俯视图 1-2），估算圆柱体断裂前的最大角速度 ω_{max}。用σ, ρ, r 来表达。

给出以下数值：$L = 1\ \mathrm{m}, r = 10\ \mathrm{cm}, \delta = 50\ \mu\mathrm{m}, \sigma = 20\ \mathrm{GPa}, \rho = 1.3 \times 9\ \mathrm{g/cm^3}$，计算$\omega_{max}$的数值。忽略形变。你可以假设侧面先于底面断裂。

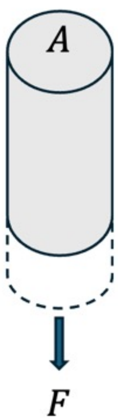
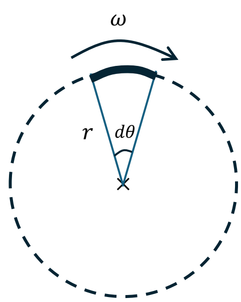


Fig 1-1



Top view
俯视图
Fig 1-2

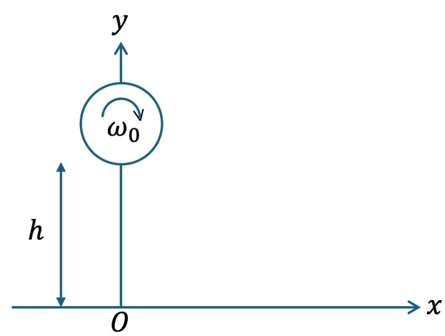


Fig 1-3

In what follows, we shall assume $\omega \ll \omega_{max}$.

(c) [2] In Fig 1-3, Alice uses a motor positioned at a height h above the ground to set the cylinder spinning with an initial angular velocity ω_0. In a prank, Bob uses his "psychic powers" to release the cylinder from the motor without altering its angular velocity. The cylinder, starting with a negligible initial translational velocity, falls toward the ground, which has a coefficient of friction μ. Ignoring air resistance, assume that collisions with the ground are perfectly elastic in the y-direction and that the collision duration t_c is negligible ($gt_c^2 \ll h$), where g represents the acceleration due to gravity. Sketch a qualitative graph representing the trajectory of the cylinder's center of mass.

在下文中，我们假设 $\omega \ll \omega_{max}$。

(c) [2] 在图 1-3，Alice 使用安装在距离地面高度为 h 处的电机，使圆柱体以初始角速度 ω_0旋转。随后 Bob 搞了个恶作剧，利用他的“超能力”使圆柱体脱离电机，且脱离过程不改变其角速度。在圆柱体的初始平动速度可忽略不计的情况下向地面下落，地面与圆柱体间的摩擦系数为 μ。忽略空气阻力，并假设：与地面的碰撞在 y 方向上为完全弹性碰撞，且碰撞时间 t_c 极短（满足 $gt_c^2 \ll h$），其中 g 为重力加速度。请画出圆柱体质心运动轨迹的定性图。

(d) The cylinder hits the ground at O at $t = 0$.
(di) [2] Find the velocity (v_x, v_y) and the angular velocity ω of the cylinder right after the 1st collision.
(dii) [3] Find the position of the ball $x(t)$ and $y(t)$, as well as the angular velocity $\omega(t)$. Express your answers in terms of the number of collisions with the ground N (We take $N = 1$ at $t = 0$), the time elapsed since the last collision Δt, h, g, r, and the ratio $K = \frac{mr^2}{I}$. You can use the floor function $\lfloor x \rfloor$ which denotes the integer part of x (e.g. $\lfloor \pi \rfloor = 3, \lfloor e \rfloor = 2$).

(d) 圆柱体在 $t = 0$ 时击中地面原点 O。

(di) [2] 求第一次碰撞后瞬间圆柱体的速度 (v_x, v_y)和角速度 ω。

(dii) [3] 求圆柱体的位置 $x(t)$和 $y(t)$，以及角速度 $\omega(t)$。用与地面碰撞的次数N（我们取 $t = 0$时 $N = 1$）、自上次碰撞以来经过的时间 Δt、h、g、r 以及比率 $K = \frac{mr^2}{I}$ 来表达你的答案。你可以使用下取整函数（floor function）$\lfloor x \rfloor$，它表示 x 的整数部分（例如 $\lfloor \pi \rfloor = 3, \lfloor e \rfloor = 2$）。