**8. JET SOUND (8 points)** — *Solution by Teo Kai Wen and Jaan Kalda.*

**i)** *(2 points)* The spectrogram shows four qualitative features.

*(a) Silence before $t_0$.* Since sound travels at a finite pace, it takes some time to hear the sound (this feature would also hold for a non-supersonic plane).

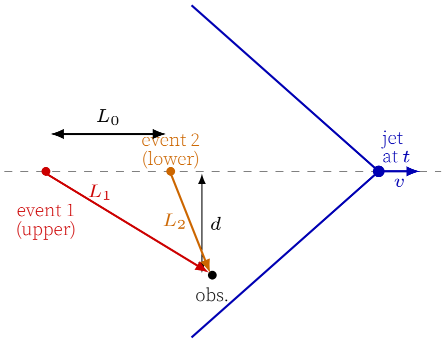
*(b) Broadband spike at $t_0$.* An entire interval of past emissions reaches the observer simultaneously, forming a shockwave front — the “boom” — that registers as a short, strong pressure peak at the sensor and represents the superposition of a wide range of frequencies.

*(c) Two simultaneous tones for $t > t_0$.* For each reception time $t > t_0$ (let us set $t = 0$), two distinct past emission events 1 and 2 contribute. Suppose the sound from event 1, emitted at time $-t_1$ from distance $L_1$, reaches us now. Since the jet is supersonic, there is a second emission — event 2 at a later time $-t_2$, distance $L_2$ from us, separated from event 1 by distance $L_0$ along the trajectory — such that

$$\frac{L_0}{v} = \frac{L_1 - L_2}{c},$$

i.e., the jet’s traversal time from 1 to 2 equals the sound delay $(L_1 - L_2)/c$. Both arrivals are Doppler-shifted, but to different degrees: the sound coming from behind (event 1, jet far upstream and approaching the observer) is blue-shifted (upper ridge), while the sound coming from ahead (event 2, jet receding past closest approach) is red-shifted (lower ridge).

*(d) Ridges decrease in time.* Both ridges diverge at the boom (the broadband spike at $t_0$) and decrease monotonically afterward, tending toward finite asymptotic values as $t \to \infty$.




**Grading.**

- Silence before $t_0$, explained as the sound not reachin the receiver yet **0.2 pts**
- Sonic boom at $t_0$, explained as the shock-wave front producing a short broadband pressure peak **0.5 pts**
- Associating the changes in frequency with the Doppler effect **0.2 pts**
- Explanation of two simultaneous tones for $t > t_0$, with two emission events reaching the observer at the same time **0.6 pts**
- Explanation of ridges diverging at the boom and tending asymptotically to finite values as $t \to \infty$ **0.5 pts**

**ii)** *(3 points)* As $t \to \infty$, the jet's velocity becomes nearly aligned with the line of sight, so the radial velocity (component along the line from source to observer) approaches $\pm v$. The classical Doppler formula for a source moving radially at speed $v_r$ relative to a stationary observer is

$$f = \frac{f_0}{|1 \pm v_r/c|},$$

with the $-$ sign for an approaching source ($v_r > 0$, source moving toward observer) and $+$ for a receding source. Applied to the two asymptotic limits with $|v_r| \to v$:

$$f_2 = \frac{f_0}{M-1} \quad \text{(upper ridge, approaching)},$$

$$f_1 = \frac{f_0}{M+1} \quad \text{(lower ridge, receding)}.$$

Combining: $\frac{1}{f_1} - \frac{1}{f_2} = \frac{2}{f_0}$ and $\frac{1}{f_1} + \frac{1}{f_2} = \frac{2M}{f_0}$, so

$$f_0 = \frac{2f_1 f_2}{f_2 - f_1}, \qquad M = \frac{f_2 + f_1}{f_2 - f_1}.$$

Reading from the spectrogram: $f_1 \approx 320\,\text{Hz}$ and $f_2 \approx 1600\,\text{Hz}$. Therefore

$$f_0 = \frac{2 \cdot 320 \cdot 1600}{1280} = 800\,\text{Hz},$$

$$M = \frac{1920}{1280} = 1.5.$$

**Grading.**

- Recognising that the radial velocity tends to $\pm v$ in the asymptotic limits **0.5 pts**
- Correct Doppler formula $f = f_0/|1 \pm v_r/c|$ **0.5 pts**
- Asymptotic upper-ridge value $f_2 = f_0/(M-1)$ **0.2 pts**
- Asymptotic lower-ridge value $f_1 = f_0/(M+1)$ **0.2 pts**
- Reading $f_1 \in [310, 350]\,\text{Hz}$ and $f_2 \in [1500, 1700]\,\text{Hz}$ from the spectrogram (0.2p each) **0.4 pts**
- Correct symbolic form $M = (f_2 + f_1)/(f_2 - f_1)$ **0.8 pts**
- Numerical answer $M \in [1.35, 1.65]$ **0.4 pts**

**iii)** *(3 points)* The closest-approach signal is the sound emitted when the jet's radial velocity is zero (jet directly overhead $B$). At that emission, $v_r = 0$, so the received frequency equals $f_0$. From part (ii)'s asymptotic relations we have

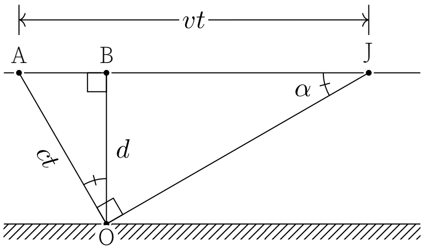
$$f_0 = \frac{2f_1 f_2}{f_2 - f_1} = \frac{2 \cdot 320 \cdot 1600}{1280} = 800\,\text{Hz}.$$

The observer hears this frequency on the *lower* ridge (the receding branch crosses $f_0$ as the jet passes overhead). Read $t_1$ from the spectrogram as the time when the lower ridge crosses $f_0 = 800\,\text{Hz}$.

Set $t = 0$ at the moment the jet emits the sound that the observer eventually hears as the boom; call this position $A$. Let $J$ denote the jet's position when the observer hears the boom, and $O$ the observer. Then

$$|AJ| = vt, \qquad |AO| = ct,$$

and the boom-tangency condition fixes $\angle AOJ = \pi/2$ (the wavefront from $A$ just grazes the observer). In particular, $\sin(\angle AJO) = |AO|/|AJ| = 1/M$, identifying $\angle AJO$ as the Mach angle $\alpha$.




Let $B$ be the foot of perpendicular from $O$ onto trajectory $AJ$, so $|OB| = d$ (closest-approach distance). The right triangles $\triangle JOA$ and $\triangle OBA$ are similar (both right-angled, sharing the angle at $A$). From this similarity:

$$|BA| = \frac{|OA|^2}{|JA|} = \frac{ct}{M},$$

$$d = \frac{|OA| \cdot |JO|}{|JA|} = \frac{ct\sqrt{M^2-1}}{M}.$$

The observer hears the boom at time $t$ and the closest-approach signal at time $|BA|/v + d/c$. The lag is

$$\tau = \frac{|BA|}{v} + \frac{d}{c} - t,$$

$$\tau = \frac{t}{M^2} + \frac{t\sqrt{M^2-1}}{M} - t.$$

Eliminating $t$ via $t = dM/(c\sqrt{M^2-1})$ and simplifying:

$$\tau c = d\left(1 - \sqrt{1-M^{-2}}\right),$$

$$\boxed{d = \frac{\tau c}{1-\sqrt{1-M^{-2}}}}.$$

Reading from the spectrogram: $t_0 \approx 3.3\,\mathrm{s}$ (boom onset) and $t_1 \approx 4.4\,\mathrm{s}$ (lower ridge crosses $800\,\mathrm{Hz}$), so $\tau \approx 1.1\,\mathrm{s}$. With $M = 1.5$:

$$1 - \sqrt{1 - 1/2.25} = 1 - \sqrt{0.5556} \approx 0.255,$$

$$d = \frac{340 \cdot 1.1}{0.255} \approx 1.5\,\mathrm{km}.$$

**Grading.**

- Computing $f_0 = 2f_1f_2/(f_2 - f_1)$ with $f_0 \in [720, 910]\,\mathrm{Hz}$ using the asymptotes from part (ii) **0.4 pts**
- Using Mach cone and the corresponding condition $\sin\alpha = 1/M$ **0.4 pts**
- Recognising that at the point of closest approach, $f = f_0$ (no Doppler shift) **0.3 pts**
- Expressing the time $t_0$ between the emission (event A) and reception of the sonic boom in terms of $d$ **0.5 pts**
- Expressing the time $t_1$ between event A and reception of the wave emitted at B in terms of $d$ **1.0 pts**
- Reading $t_0, t_1$ from the spectrogram and computing $\tau \in [0.85, 1.15]\,\mathrm{s}$ **0.2 pts**
- Numerical answer if approach correct $d \in [1.0, 1.7]\,\mathrm{km}$ **0.2 pts**