7. Water hose (5 points) — *Solution by Ralf Robert Paabo and Jaan Kalda.*

i) *(1.5 points)* **Solution 1 (direct from the problem's drag scales).** The problem gives the laminar drag per unit area as $F_L \sim \mu v/D$ (the velocity gradient at the wall is of order v/D). The turbulent drag per unit area is $F_T \sim \rho v^2$ (dynamic pressure carried by the vortices). The dimensionless ratio is

$$R = \frac{F_T}{F_L} \sim \frac{\rho v^2}{\mu v/D} = \frac{\rho v D}{\mu}.$$

. Some solutions tried to add numerical factors, however, it is an impossible task, because drag force in turbulent case is chaotic. Therefore any numerical factor calculated is useless. However, no points are deducted unless the factor is very big (> 10). A very big factor would lead to inaccuracies when using R to determine flow type.

Solution 2 (dimensional analysis). Look for $R = D^d \rho^r \mu^m v^u$ to be dimensionless. Mass, length, and time give three equations:

$$\begin{cases} r + m = 0, \\ d - 3r - m + u = 0, \\ -m - u = 0. \end{cases}$$

Solving: $m = -r$, $u = r$, $d = r$. So $R = (Dv\rho/\mu)^r$. This seemingly leaves one degree of freedom, but μ appears only in the laminar drag (denominator of F_T/F_L), so $m = -1$ and hence $r = 1$:

$$R = \frac{\rho v D}{\mu}.$$

Grading (Solution 1).

- $R = \frac{F_T}{F_L} \sim \frac{\rho v^2}{\mu du/dr}$ **0.5 pts**
- $du/dr \sim v/D$ **0.5 pts**
- $R = \rho v D/\mu$ **0.5 pts**
- Includes numerical factor > 10 **−0.2 pts**

Grading (Solution 2).

- Setting up $R = D^d \rho^r \mu^m v^u$ as dimensionless **0.3 pts**
- Correct system of equations from mass, length, time dimensions **0.5 pts**
- Solving the system to obtain $R = (Dv\rho/\mu)^r$ **0.4 pts**
- Correctly fixing $r = 1$ by noting μ enters with power -1 (or by appeal to the standard Reynolds-number form) **0.3 pts**

ii) *(0.5 points)* The hose has cross-section area $\pi d^2/4$, so by continuity the mean flow speed is

$$v = \frac{Q}{\pi d^2/4} = \frac{4Q}{\pi d^2} \approx 3.1\,\frac{\text{m}}{\text{s}}.$$

The Reynolds number is then

$$R = \frac{\rho v d}{\mu} = \frac{4\rho Q}{\pi d \mu} \approx 37000.$$

Since $37000 \gg 2500$, the flow is *turbulent*.

Grading.

- Computing the mean flow speed v from Q via continuity **0.2 pts**
- Correctly evaluating $R \approx 37000$ **0.2 pts**
- Concluding turbulent flow on the basis of the numerical comparison $R \gg 2500$ **0.1 pts**

iii) *(3 points)* The pump's power P goes into three reservoirs: gravitational potential energy of the lifted water, work against drag in the hose, and kinetic energy of the exiting jet. Per unit volume, these are ρgh, Δp_{drag}, and $\rho v_{\text{exit}}^2/2$. The power balance is therefore

$$P = Q\left(\rho gh + \Delta p_{\text{drag}} + \tfrac{1}{2}\rho v_{\text{exit}}^2\right).$$

Since the flow is turbulent (part ii), the drag scales as v^2: $\Delta p_{\text{drag}} = kv^2$, where k is a constant for this hose.

Calibration (no nozzle). With no nozzle, $v_{\text{exit}} = v_0$ (the hose-exit speed from part ii, $v_0 \approx 3.14\ \frac{\text{m}}{\text{s}}$). Then

$$P = Q\left(\rho gh + kv_0^2 + \tfrac{1}{2}\rho v_0^2\right).$$

Numerically:

$$\rho gh = 1.96 \times 10^5\ \text{Pa}, \quad \tfrac{1}{2}\rho v_0^2 = 4.93 \times 10^3\ \text{Pa},$$

$$P/Q - \rho gh - \tfrac{1}{2}\rho v_0^2 = kv_0^2 \approx 3.99 \times 10^5\ \text{Pa},$$

$$k = \frac{3.99 \times 10^5\ \text{Pa}}{v_0^2} \approx 4.04 \times 10^4\ \frac{\text{Pa}\cdot\text{s}^2}{\text{m}^2}.$$

With nozzle. The exit area is f times the hose area; by mass continuity, $v'_{\text{exit}} = v'/f$, where v' is the (new) hose flow speed. Substituting into the power balance:

$$P = \frac{\pi d^2}{4} v' \left(\rho gh + kv'^2 + \frac{\rho v'^2}{2f^2}\right).$$

This is a cubic in v'.

Dimensionless form. Introduce $x = v'/v_0$. Each of the three reservoirs in the unblocked case contributes a fraction of the total power:

$$a = \frac{}{\rho gh + kv_0^2 + \frac{1}{2}\rho v_0^2}, \quad b = \frac{{}^{2}_{0}}{\rho gh + kv_0^2 + \frac{1}{2}\rho v_0^2}$$

and $c = 1 - a - b$. Numerically: $a \approx 0.327$, $b \approx 0.665$, $c \approx 0.0082$. The power balance becomes

$$x\left[a + \left(b + \tfrac{c}{f^2}\right)x^2\right] = 1.$$

With $f = 0.15$, $c/f^2 \approx 0.364$, so $b + c/f^2 \approx 1.029$:

$$x^3 + 0.318\,x = 0.972 \Rightarrow x = (0.972 - 0.318\,x)^{1/}$$

Iteration. Starting from $x_0 = 1$ on the right-hand side:

$$x_1 = (0.972 - 0.318)^{1/3} = 0.868,$$

$$x_2 = (0.972 - 0.276)^{1/3} = 0.886,$$

$$x_3 = (0.972 - 0.282)^{1/3} = 0.884,$$

$$x_4 = (0.972 - 0.281)^{1/3} = 0.884.$$

Converged: $x \approx 0.884$.

Results.

$$Q'/Q = v'/v_0 = x \approx \boxed{0.88}, \quad \text{so} \quad Q' \approx 22.1\,\frac{}{\mathrm{m}}$$

The exit speed is $v'_{\text{exit}} = v'/f = xv_0/f$, and the dirty-surface excess pressure scales as $\rho v_{\text{exit}}^2/2$:

$$\frac{p'_{\text{excess}}}{p_{\text{excess}}} = \frac{v'^2_{\text{exit}}}{v_0^2} = \frac{x^2}{f^2} = \frac{0.884^2}{0.15^2} \approx \boxed{35}.$$

Grading.

- Power balance: gravity term $\rho g h$ **0.2 pts**
- Power balance: drag term Δp_{drag} **0.2 pts**
- Power balance: exit kinetic energy term $\rho v_{\text{exit}}^2/2$ **0.2 pts**
- Identifying $\Delta p_{\text{drag}} = k v^2$ from the turbulent-regime scaling **0.3 pts**
- Calibrating k from the unblocked case: $k \approx 4 \times 10^4\ \frac{\mathrm{Pa \cdot s^2}}{\mathrm{m^2}}$ **0.3 pts**
- Mass continuity giving $Q_m = Q'_m$ **0.3 pts**
- $v'_{\text{exit}} = v'/f$ **0.2 pts**
- Updated power balance with the kinetic-energy term boosted by $1/f^2$ **0.5 pts**
- Volumetric flow rate ratio $Q'/Q \approx 0.88$ within tolerance (± 0.02) **0.4 pts**
- Formula for excess-pressure ratio $\left(v'_{\text{exit}}/v_0\right)^2$ **0.2 pts**
- Excess-pressure ratio ≈ 35 within tolerance (± 1) **0.2 pts**
- If accuracy > 1 % **−0.1 pts**