6. Pull-up bar rope (5 points) — *Solution by Jaan Kalda.*

i) *(1 point)* Place the bar at the origin of a coordinate system where x is the horizontal distance from the bar to the weight along the floor, and H is the height of the bar above the floor. The length of the bar-to-weight rope segment is

$$L = \sqrt{x^2 + H^2}, \qquad \sin\alpha = x/L.$$

Method 1 (inextensibility / velocity projection). Since the rope is inextensible, the velocity components of its two endpoints *along the rope direction* must match: whatever rate one end recedes from a point on the rope, the other end must approach it at the same rate. The puller's hand pulls the rope

along its own direction at speed v, so the rope shortens at rate v. The weight moves horizontally toward the bar at speed u; its velocity component along the rope (which makes angle α with the vertical, hence angle $\frac{\pi}{2} - \alpha$ with the floor) is $u\sin\alpha$. Equating:

$$u\sin\alpha = v \quad\Longrightarrow\quad \boxed{u = \frac{v}{\sin\alpha}}.$$

Method 2 (implicit differentiation). Differentiating $L^2 = x^2 + H^2$ with respect to time:

$$L\dot{L} = x\dot{x}.$$

With $\dot{L} = -v$ (the segment shortens at constant rate) and $\sin\alpha = x/L$:

$$\dot{x} = -\frac{Lv}{x} = -\frac{v}{\sin\alpha},$$

$$u = |\dot{x}| = \frac{v}{\sin\alpha}.$$

Grading (Method 1).

- Recognising that the velocity components of the rope endpoints along the rope direction must match (rope inextensibility) **0.4 pts**
- Identifying the projection $u\sin\alpha$ for the weight's velocity along the rope **0.3 pts**
- Final answer $u = v/\sin\alpha$ **0.3 pts**

Grading (Method 2).

- Setting up $L^2 = x^2 + H^2$ with $\dot{L} = -v$ **0.3 pts**
- Correctly differentiating to obtain $L\dot{L} = x\dot{x}$ **0.4 pts**
- Final answer $u = v/\sin\alpha$ **0.3 pts**

ii) *(1 point)* **Method 1 (continuing from $L\dot{L} = x\dot{x}$).** Differentiate again, with $\dot{L} = -v$ constant (so $\ddot{L} = 0$):

$$\dot{L}^2 = \dot{x}^2 + x\ddot{x},$$

$$\ddot{x} = \frac{v^2 - \dot{x}^2}{x} = \frac{v^2 - u^2}{x}.$$

Since $u^2 > v^2$, we have $\ddot{x} < 0$: the weight accelerates toward the bar. With $u = v/\sin\alpha$ and $x = H\tan\alpha$:

$$|\ddot{x}| = \frac{u^2 - v^2}{x} = \frac{v^2(1 - \sin^2\alpha)/\sin^2\alpha}{H\tan\alpha},$$

$$|\ddot{x}| = \boxed{\frac{v^2}{H}\cot^3\alpha}.$$

- Correctly differentiating an incorrect result from the previous part **0.5 pts**.

Method 2 (direct differentiation of $u = v/\sin\alpha$). The angle α changes in time as the weight slides; introduce $\omega = -\dot{\alpha}$ (positive as α decreases). The weight's position is $x = H\tan\alpha$, so its speed is

$$u = -\dot{x} = -\frac{H}{\cos^2\alpha}\,\dot{\alpha} = \frac{H\omega}{\cos^2\alpha}.$$

Combined with $u = v/\sin\alpha$:

$$\omega = \frac{u\cos^2\alpha}{H} = \frac{v\cos^2\alpha}{H\sin\alpha}.$$

Differentiating $u = v/\sin\alpha$ with respect to time:

$$\dot{u} = -\frac{v\cos\alpha}{\sin^2\alpha}\,\dot{\alpha} = \frac{v\omega\cos\alpha}{\sin^2\alpha}.$$

Substituting ω:

$$|\ddot{x}| = \dot{u} = \frac{v\cos\alpha}{\sin^2\alpha}\cdot\frac{v\cos^2\alpha}{H\sin\alpha} = \boxed{\frac{v^2}{H}\cot^3\alpha}.$$

Grading (Method 1).

- Differentiating $L\dot{L} = x\dot{x}$ to obtain $\ddot{x} = (v^2 - u^2)/x$ **0.5 pts**
- Substituting u and x to obtain final answer $a = (v^2/H)\cot^3\alpha$ **0.5 pts**

Grading (Method 2).

- Introducing $\omega = -\dot{\alpha}$ as the angular rate of the rope direction **0.2 pts**
- Relating the weight's speed to ω: $u = H\omega/\cos^2\alpha$ **0.2 pts**
- Deriving $\omega = v\cos^2\alpha/(H\sin\alpha)$ from $u = v/\sin\alpha$ **0.2 pts**
- Differentiating $u = v/\sin\alpha$ to obtain $\dot{u} = v\omega\cos\alpha/\sin^2\alpha$ **0.2 pts**
- Substituting ω and obtaining final answer $a = (v^2/H)\cot^3\alpha$ **0.2 pts**

iii) *(3 points)* While the weight is on the floor, the only horizontal force on it is the horizontal component of the rope tension $T\sin\alpha$, directed toward the bar. Newton's second law in the horizontal direction:

$$T \sin\alpha = M|\ddot{x}| = \frac{Mv^2\cos^3\alpha}{H\sin^3\alpha},$$

$$T = \frac{Mv^2\cos^3\alpha}{H\sin^4\alpha}.$$

Vertically, $T\cos\alpha + N = Mg$, where N is the normal force from the floor. The weight lifts off when $N = 0$, i.e., $T\cos\alpha_0 = Mg$:

$$\frac{Mv^2\cos^4\alpha_0}{H\sin^4\alpha_0} = Mg,$$

$$\tan^4\alpha_0 = \frac{v^2}{gH},$$

$$\boxed{\alpha_0 = \arctan\sqrt[4]{\frac{v^2}{gH}}}.$$

Grading.

- Identifying that the only horizontal force on the weight is $T\sin\alpha$ **0.3 pts**
- Writing Newton's second law horizontally: $T\sin\alpha = M|\ddot{x}|$ **0.3 pts**
- Substituting the expression for $|\ddot{x}|$ from part (ii) **0.2 pts**
- Isolating T to obtain $T = Mv^2\cos^3\alpha/(H\sin^4\alpha)$ **0.2 pts**
- Correctly setting up the vertical force balance $T\cos\alpha + N = Mg$ **0.5 pts**
- Correct lift-off condition $N = 0$, i.e., $T\cos\alpha_0 = Mg$ **0.5 pts**
- Algebraic manipulation to $\tan^4\alpha_0 = v^2/(gH)$ **0.5 pts**
- Final answer $\alpha_0 = \arctan\sqrt[4]{v^2/(gH)}$ **0.5 pts**