3. Kirill on a swing (8 points) — *Solution by Kaarel Hänni.*

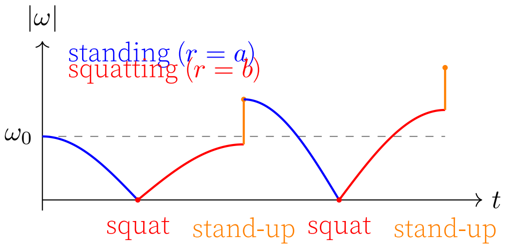
i) *(1.5 points)* Starting at the bottom with $|\omega| = \omega_0$, Kirill (standing, radius a) swings up; $|\omega|$

falls smoothly to 0 at the turning point, after a quarter-period $T_a/4 = (\pi/2)\sqrt{a/g}$. At the turning point Kirill squats (radius $a \to b$, no effect on the plot since $\omega = 0$). Swinging back while squatting, $|\omega|$ rises smoothly from 0 to η_1 (see part c; $\eta_1 = \omega_0\sqrt{a/b} < \omega_0$) after a quarter-period $T_b/4 = (\pi/2)\sqrt{b/g}$, which is *longer* than the first since $b > a$. At the bottom Kirill stands up: $|\omega|$ *jumps* from η_1 to $\omega_1 = \eta_1(b/a)^2 = \omega_0(b/a)^{3/2} > \omega_0$. The second half of the period repeats the pattern with new amplitudes ω_1, $\eta_2 = \omega_1\sqrt{a/b} > \omega_0$, $\omega_2 = \omega_0(b/a)^3$.

The sketch should therefore show:

- smooth arches during swinging, cusps at the bottom (jumps) and smoothness at turning points;
- the "squatting" arches taking longer than the "standing" ones;
- growth of successive peaks.

Part (i): Angular speed vs time, first full period.



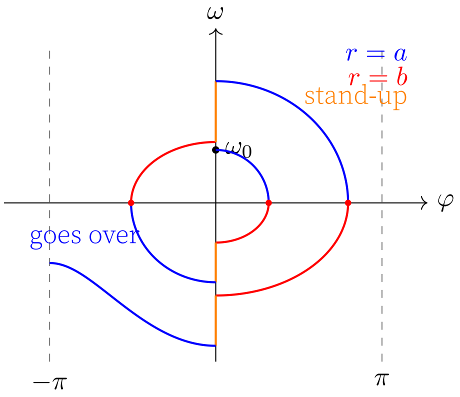
Part (i) — qualitative sketch of $|\omega|$ (or signed ω) vs t:

- Approximately sinusoidal arcs between jumps **0.3 pts**
- Instantaneous jumps at the maximum values of ω (at the bottom of the swing, during stand-up) **0.6 pts**
- Smaller amplitude during "squatting" arches, or equivalently a longer quarter-period while $r = b$ **0.3 pts**
- Overall growth of peak ω over the period **0.3 pts**

A sketch of signed ω (rather than $|\omega|$) receives full credit if the curve correctly corresponds to the axis labels and all other qualitative features are preserved. Failure to correctly mark $|\omega|$, where necessary, deducts **0.5 pts**. The same goes for not completing the full period of the first swing.

ii) *(2 points)* The trajectory in the (α, ω) plane pieces together arcs of pendulum-like phase curves at two radii, a and b. Each half-swing traces an arc of the energy conservation curve for the relevant radius. Stand-ups at $\alpha = 0$ appear as *vertical segments* where $|\omega|$ jumps from η_i to $\omega_i = \eta_i(b/a)^2$. Squats at the turning points ($\omega = 0$) appear as smooth transitions (the radius-a arc meets the radius-b arc tangentially at $\omega = 0$).

The motion is an outward spiral: each full cycle brings Kirill back to $\alpha = 0$ with a larger $|\omega|$. When ω at the bottom (standing, radius a) first exceeds $\omega_{\min} = 2\sqrt{g/a}$, the next upward arc does not close: it reaches $\alpha = \pi$ with $\omega > 0$ and continues past the top. This is the open curve on the sketch. **Part (ii): Phase diagram (ω vs φ), first three half-swings.**



Key features for grading:

- lens-shaped/pendulum-like arcs (not ellipses);
- vertical line segments only at $\alpha = 0$ (stand-ups), smooth meeting at turning points (squats);
- spiral grows outward;
- final open curve goes over the top with non-zero ω.

Failure to use the phase diagram of angular velocity and angle, deducts **0.5 pts** from the sum of part ii).

The number of spiral loops is not required to match the actual answer N computed in part (c).

Part (ii) — qualitative sketch of phase diagram ω vs φ:

- Elliptical arcs for small-amplitude (early) cycles, with standing arcs ($r = a$) visibly taller (steeper in ω) than squatting arcs ($r = b$) **0.5 pts**
- Trajectory passes *continuously* through the turning points ($\omega = 0$): no jump at maximum $|\varphi|$ **0.3 pts**
- Vertical jumps at $\varphi = 0$ (during stand-up at the bottom): ω jumps by factor $(b/a)^2$ **0.4 pts**
- Overall growth: each successive cycle encloses the previous one, spiralling outward **0.4 pts**
- Final half-cycle goes over the top — trajectory crosses $\varphi = \pm\pi$ with ω still nonzero (no turning point on this segment) **0.4 pts**

iii) *(4.5 points)* **Growth factor per stand-up.** Let ω_i denote the angular velocity just after the i-th stand-up (at the bottom, standing, radius a); let η_i denote the angular velocity just before that stand-up (at the bottom, squatting, radius b).

Energy conservation on the way up (standing, radius a) to turning point at angle α:

$\frac{1}{2}(\omega_i a)^2 = ga(1-\cos\alpha) \implies \omega_i^2 a = 2g(1-\cos\alpha).$

At the turning point $\omega = 0$, so the squat ($a \to b$) leaves α and ω unchanged. Swinging back down (squatting, radius b):

$$\frac{1}{2}(\eta_{i+1} b)^2 = gb(1-\cos\alpha)$$
$$\implies \eta_{i+1}^2 b = 2g(1-\cos\alpha).$$

Equating: $\eta_{i+1} = \omega_i\sqrt{a/b}$. The stand-up is a purely radial motion, so angular momentum about the pivot is conserved: $\eta_{i+1} b^2 = \omega_{i+1} a^2$, whence

$$\frac{\omega_{i+1}}{\omega_i} = \sqrt{\frac{a}{b}} \cdot \frac{b^2}{a^2} = \left(\frac{b}{a}\right)^{3/2}.$$

After N stand-ups, $\omega_N = \omega_0 (b/a)^{3N/2}$.

Minimum angular velocity for a loop. The rods are rigid, so the condition for reaching the top is simply that the kinetic energy at the bottom suffice to raise Kirill by $2a$:

$$\frac{1}{2}(\omega_{\min}a)^2 = 2ga \implies \omega_{\min} = 2\sqrt{g/a}.$$

Counting. Kirill first loops over when $\omega_N \geq \omega_{\min}$, i.e.

$$N \geq \frac{2}{3}\frac{\ln(\omega_{\min}/\omega_0)}{\ln(b/a)}.$$

Numerical evaluation. With $a = 2.5$ m, $b = 3.0$ m, $\omega_0 = 1.0$ rad/s, $g = 10$ m/s^2:

$$\begin{aligned} b/a &= 1.2, \quad (b/a)^{3/2} \approx 1.3145, \\ \omega_{\min} &= 2\sqrt{10/2.5} = 4\ \text{rad/s}, \\ N &\geq \frac{2}{3}\frac{\ln 4}{\ln 1.2} = \frac{2}{3}\cdot\frac{1.386}{0.1823} \approx 5.07 \\ &\implies \boxed{N = 6}. \end{aligned}$$

Check: $\omega_5 = 1.2^{7.5} \approx 3.93$ rad/s (just short); $\omega_6 = 1.2^9 \approx 5.16$ rad/s (exceeds $\omega_{\min}$). So Kirill loops over on the swing-up following his *sixth* stand-up.

Grading (part c):

- Angular momentum conservation during the stand-up **0.7 pts**
- Energy conservation during the swing **0.7 pts**
- Growth factor $(b/a)^{3/2}$ per stand-up **0.7 pts**
- Minimum angular velocity $\omega_{\min}^2 = 4g/a$ for the rigid-rod swing to loop over the top **0.8 pts**
- Correct inequality for the over-the-top condition ($\omega_N \geq \omega_{\min}$ after the Nth stand-up) **0.4 pts**
- Solving the inequality for N using the growth law $\omega_N = \omega_0(b/a)^{3N/2}$, yielding $N \geq \log[4g/(a\omega_0^2)]/[3\log(b/a)]$ **0.7 pts**
- Numerical answer $N = 6$ **0.5 pts**