T2: Hysteresis (10 pts)

a) Solution considering remagnetization time

The current $I(t)$ in the circuit and the voltages $U_L(t)$, $U_C(t)$ across the solenoid and the capacitor are connected through equations

$$\begin{aligned} I(t) &= C\frac{\mathrm{d}U_C}{\mathrm{d}t} = -C\frac{\mathrm{d}U_L}{\mathrm{d}t} \\ U_L(t) &= NA\frac{\mathrm{d}B}{\mathrm{d}t} = -U_C(t) \end{aligned} \tag{3}$$

where $A = \pi R^2$. The magnetic field strength in the solenoid is

$$H = \frac{NI}{\ell}\,. \tag{4}$$

At the start there is a large current I_i in the circuit such that $H(I_\mathrm{i}) \gg H_0$. Initially, the current charges the capacitor, while the current $I(t)$ and the field $H(t)$ decrease. The point, representing the momentary combination of $H(t)$ and $M(t)$ moves along the top horizontal line in the hysteresis curve (to the left). The magnetization M does not change during this phase even when I crosses 0 and becomes negative. In this phase $\frac{\mathrm{d}B}{\mathrm{d}t} = \mu_0\frac{\mathrm{d}H}{\mathrm{d}t}$. As in an ideal LC-circuit, energy is conserved.

When the current reaches the critical magnitude in the opposite direction, i.e.

$$I = -I_0 = -\frac{H_0\ell}{N}\,, \tag{5}$$

the magnetization M starts to reverse. The vertical hysteresis curve at H_0 means that during the magnetization reversal (which takes some time) H stays constant and equal to H_0. Therefore, also the current stays constant and equal to $-I_0$, while M changes. For the critical current we obtain

$$-I_0 = C\frac{\mathrm{d}U_C}{\mathrm{d}t} \tag{6}$$

$$U_C(t) = -NA\mu_0\frac{\mathrm{d}M}{\mathrm{d}t} \tag{7}$$

Solving these equations with $t = 0$ corresponding to the moment I reaches $-I_0$ (when the orientation of magnetization in the core starts to change) and using $U_1 = U_C(0)$ gives

$$U_C(t) = U_1 - \frac{I_0}{C}t \tag{8}$$

$$M(t) = M_0 - \frac{1}{NA\mu_0}\left(U_1 t - \frac{I_0}{2C}t^2\right) \tag{9}$$

After the time interval Δt the core is completely remagnetized with $M(\Delta t) = -M_0$. We get

$$\Delta t = \frac{C}{I_0}\left(U_1 - \sqrt{U_1^2 - \frac{4NA\mu_0 I_0 M_0}{C}}\right) \tag{10}$$

The corresponding voltage is

$$U_2 = U_C(\Delta t) = \sqrt{U_1^2 - \frac{4NA\mu_0 I_0 M_0}{C}}. \tag{11}$$

The energy needed to change the orientation of magnetization of the core thus is

$$\frac{1}{2}C(U_1^2 - U_2^2) = 2NA\mu_0 I_0 M_0 = 2A\ell\mu_0 H_0 M_0 \tag{12}$$

After complete remagnetization the circuit behaves again like an LC-circuit, but with lower energy. From Δt on the current changes harmonically until it reaches the critical value I_0, and the story repeats. Since the energy needed for the magnetization reversal process is independent of the starting voltage (as long as it is fully completed), during one full cycle the energy loss is twice the value computed above, i.e. $\Delta E = 4A\ell\mu_0 H_0 M_0$. The maximum current I_2 after one cycle is reduced from I_i due to the lower energy with $\Delta E = \frac{1}{2}L(I_\mathrm{i}^2 - I_2^2)$, where L is the inductance $L = \mu_0 N^2 A/\ell$ of the solenoid. It follows

$$I_2 = \sqrt{I_\mathrm{i}^2 - 8\frac{A}{L}\ell\mu_0 H_0 M_0} = I_\mathrm{i}\sqrt{1 - \frac{8\ell^2 H_0 M_0}{N^2 I_\mathrm{i}^2}} \tag{13}$$

Note that $NI_\mathrm{i}/\ell = H(I_\mathrm{i})$. Since $H(I_\mathrm{i}) \gg H_0$ we can approximate the change ΔI in the maximum current to

$$\begin{aligned} \Delta I = I_2 - I_\mathrm{i} &= I_\mathrm{i}\sqrt{1 - \frac{8\ell^2 H_0 M_0}{N^2 I_\mathrm{i}^2}} - I_\mathrm{i} \\ &\approx -\frac{4\ell^2 H_0 M_0}{N^2 I_\mathrm{i}} = -4I_\mathrm{i}\frac{H_0 M_0}{H(I_\mathrm{i})^2} \end{aligned} \tag{14}$$

Solution using energy considerations

Originally the energy used to remagnetize the core is stored in the inductor, so $E_0 = \frac{1}{2}LI_\mathrm{i}^2$. After a single cycle, the energy is $E_2 = \frac{1}{2}LI_2^2$, where the same notation as in the other approach is used. The inductance of the solenoid is given by $L = \mu_0 N^2 A/\ell$. The loss of stored energy during one cycle is proportional to the area of the square in the hysteresis curve (an explanation is provided below) with $\Delta E = 4A\ell\mu_0 H_0 M_0$.

We thus get

$$I_2 = I_\mathrm{i}\sqrt{1 - \frac{8\ell^2 H_0 M_0}{N^2 I_\mathrm{i}^2}} \tag{15}$$

from which the result (14) follows as above.

To understand the relation between the energy loss during magnetization and the area of the square in the hysteresis curve, we consider the instantaneous electrical power in the solenoid and integrate it over over one cycle. Using equations (3) and (4) we have

$$\begin{aligned} P = U_L I &= ANI\frac{\mathrm{d}B}{\mathrm{d}t} = AH\ell\frac{\mathrm{d}B}{\mathrm{d}t} \\ &= \mu_0 AH\ell\left(\frac{\mathrm{d}H}{\mathrm{d}t} + \frac{\mathrm{d}M}{\mathrm{d}t}\right) \end{aligned} \tag{16}$$

Since the circuit is isolated and the capacitor uncharged at the beginning of an oscillation cycle the total electrical work over one cycle, starting at t_i and ending at t_e should equal zero. Integrating the power over the cycle gives

$$\begin{aligned} 0 &= \int_{t_\mathrm{i}}^{t_\mathrm{e}} P(t)\mathrm{d}t \\ &= \mu_0 A\ell \left[\int_{t_\mathrm{i}}^{t_\mathrm{e}} H(t)\frac{\mathrm{d}H}{\mathrm{d}t}\mathrm{d}t + \int_{t_\mathrm{i}}^{t_\mathrm{e}} H(t)\frac{\mathrm{d}M}{\mathrm{d}t}\mathrm{d}t \right] \end{aligned} \tag{17}$$

The first integral gives $\frac{1}{2}H(t_\mathrm{e})^2 - \frac{1}{2}H(t_\mathrm{i})^2$. The second integral gives $4H_0M_0$, from which we get

$$\Delta E = \frac{1}{2}L\left(I(t_\mathrm{i})^2 - I(t_\mathrm{e})^2\right) = 4A\ell\mu_0 H_0 M_0 \tag{18}$$

Note: The quantity $\mathrm{d}u = H\mathrm{d}B$ is the change per unit volume of the magnetic energy of the solenoid.

b) As long as the current amplitude exceeds the critical current required to change the magnetization orientation, the circuit will continue loosing energy in the remagnetization process. After many cycles the voltage at the start of the remagnetization process is too small for the remagnetization to be fully completed. The remagnetization process during the phase of the constant current I_0 stops when the voltage (and charge in capacitor) becomes zero. In this case no more energy is stored in the capacitor that can be used for remagnetization. The core then remains partially magnetized and there is no more energy dissipation connected with the core.

From now on, the circuit behaves like an LC circuit with current amplitude of $I_0 = \frac{H_0\ell}{N}$. In the $H - M$ graph this corresponds to moving back and forth on a horizontal line within the square. Therefore, the maximum current after many oscillations is the amplitude of the current oscillations in the LC-circuit

$$I_\mathrm{max} = I_0 = H_0\frac{\ell}{N}. \tag{19}$$

c) The two distinct phases are the LC-circuit harmonic behaviour (phase A) and the remagnetization process (phase B). To find the maximum duration for the latter let us first consider the duration of the magnetization processes for the case where full magnetization occurs. The duration was computed in (10) and equals

$$\Delta t = \frac{C}{I_0}\left(U_1 - \sqrt{U_1^2 - \frac{4NA\mu_0 I_0 M_0}{C}}\right) \tag{20}$$

Since I_0 is the same and constant for each magnetization process, we have to find the minimal value of Δt over all possible U_1. We can compute a derivative of Δt with respect to U_1,

$$\frac{\mathrm{d}\Delta t}{\mathrm{d}U_1} = \frac{C}{I_0}\left(1 - \frac{U_1}{\sqrt{U_1^2 - 4NA\mu_0 I_0 M_0/C}}\right) < 0. \quad (21)$$

The derivative is negative for all values of U_1, which means that Δt is monotoneously decreasing with increasing U_1. Thus the voltage has to be as small as possible. On the other hand, when the voltage at the start of the cycle is too small, it drops to zero before the core is fully remagnetized. It follows from equation (6) that this happens after time

$$\Delta t = \frac{U_1 C}{I_0} \quad (22)$$

Hence for maximal duration Δt the voltage U_1 should be as large as possible. Together, the optimal initial voltage is the one where the voltage drops to zero precisely in the moment when the core is fully magnetized. The corresponding U_1 can be found for instance using (10) as the smallest U_1 for which a real-valued Δt exists, which is

$$U_1 = 2\sqrt{\frac{NA\mu_0 I_0 M_0}{C}} \quad (23)$$

This gives the maximum duration as

$$\begin{aligned} \Delta t_{\text{max}} &= 2\sqrt{\frac{NA\mu_0 C M_0}{I_0}} \\ &= 2\sqrt{\frac{N^2 A\mu_0 C M_0}{\ell H_0}} = 2\sqrt{LC\frac{M_0}{H_0}} \end{aligned} \quad (24)$$

Marking scheme T2

Problem T2 a)		Pts.
A	Relate energy loss to capacitor voltage drop (12) *or* area under H-M-curve. Stating only $E_c = \frac{1}{2}CU^2$ gives 0.3 pts	0.4
B	Determine energy loss over full cycle from capacitor voltage drop *or* area under H-M-curve (-0.4 pts if only half cycle is considered or numerical factor is erroneous)	1.0
C	Relate current to energy $\frac{1}{2}LI^2$ stored in inductor	0.3
D	State/Derive/Use $L = \mu_0 N^2 A/\ell$ for inductance of solenoid	0.3
E	Determine change in current (14) with or without approximation	1.0
Partial sum T2 a)		**3.0**

Problem T2 b)		**Pts.**
A	State/Derive/Use $H(I) = \frac{NI}{\ell}$	0.3
B	Realize LC-circuit behaviour, when no remagnetization occurs	0.5
C	Realize that energy is lost only in remagnetization	0.5
D	Determine critical current I_0 (19)	0.5
E	Realize that current amplitude after many oscillations equals critical current	0.5
Partial sum T2 b)		**2.3**

Problem T2 c)		**Pts.**
A	Identify phase B as remagnetization process	0.3
B	State current and voltage equations (3) (0.3 pts. each, only 0.1 pts. each for $Q = C/U$ and $U = \mathrm{d}\phi/\mathrm{d}t$)	0.6
C	Realize that current stays at critical current I_0 during remagnetization	0.5
D	Determine M as function of time (9)	0.5
E	Determine time interval for remagnetization (10)	0.5
F	For complete remagnetization - Derive that duration is maximised for minimal possible voltage	0.5
G	For incomplete remagnetization - Determine time for complete discharge of capacitor	0.5
H	For incomplete remagnetization - Derive that duration is maximised for maximal possible voltage	0.3
I	Compute optimal value for voltage (23)	0.5
J	Determine maximum duration (24)	0.5
Partial sum T2 c)		**4.7**

General rules for marking in T2:

- **Depending on the approach taken, points may be awarded also for subquestions not directly.** So make sure to check all aspects of the marking scheme even if students only worked e.g. on part a).
- The grain size for marking is 0.1 Pts.
- Partial marks can be awarded for most aspects.
- For each mistake in calculation (algebraic or numeric) 0.2 Pts. are deducted.
- If a mistake leads to a dimensionally incorrect expression no marks are given for the result.
- Propagating errors are not punished again unless they are dimensionally wrong or entail oversimplified/wrong physics (e.g. using an approximation in a regime it does not apply).